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ON FORMULAS AND METHODS FOR EXPRESSING THE ATTITUDE OF A RIGID BODY

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1. Introduction

In many fields of classical mechanics there is the need of specifying the attitude of a rigid body. Classical examples can be found in the dynamic analysis of gyroscopes, spatial mechanisms. In computer graphics the need of simulating the movement of objects, or the point of view displacement, requires a mathematical tool for the representation of the attitude of bodies. The purpose of this paper is somewhat tutorial. However, it includes some *ad hoc* derived analytical expressions.

2. Mathematical preliminaries

The attitude of a body relative to another is mathematically defined by a coordinate transformation between reference frames attached to each body. Although described through a matrix transformation, it is necessary to distinguish between *transition* and *rotation*. In particular, with reference to the schemes presented in Figure 1:

- *transisition* is the process of changing, for the same vector, the attitude of a coordinate systems;
- *rotation* is the process of changing the attitude of a vector within the same coordinate system.

In order to transit from the initial coordinate system (labelled with subscript 0) to the final coordinate system (labelled with subscript 1), we write:

$$\{r_1\} = [T]_0^1 \begin{Bmatrix} x_0 \\ y_0 \\ z_0 \end{Bmatrix} \quad (1), \text{ where } \{r_1\} = \{x_1 \quad y_1 \quad z_1\}^T \text{ and } \{r_0\} = \{x_0 \quad y_0 \quad z_0\}^T .$$

In order to express the new coordinates (labelled with subscript 1) of the rotated vector in terms of the old (labelled with subscript 0) of the same vector prior to rotation, we write:
$\{r_1\} = [R]_0^1 \{r_0\}$ (2). Spherical motions are usually described by means of the previous tranformation. The following relation $[R^T]_0^1 = [T]_0^1$ (3), can be obtained through inspection. Since all the transformations introduced preserve the rigidity, matrix $[R]$ is orthonormal (*i.e.* $[R]^T[R] = [R][R]^T = [I]$).

In all the subsequent algebraic treatment the cartesian systems are assumed right-handed. Matrices are not the only mathematical tools for expressing rotations or change of coordinates. However, for space reasons we limit this review to matrix formulas only.

3. Review of methods

In general a minumum of three parameters are required to describe the attitude of a body fixed coordinate system $o-xyz$ w.r.t. an inertial $O-XYZ$ coordinate system. However, in some cases, redundancy can be used to avoid undeterminacy.

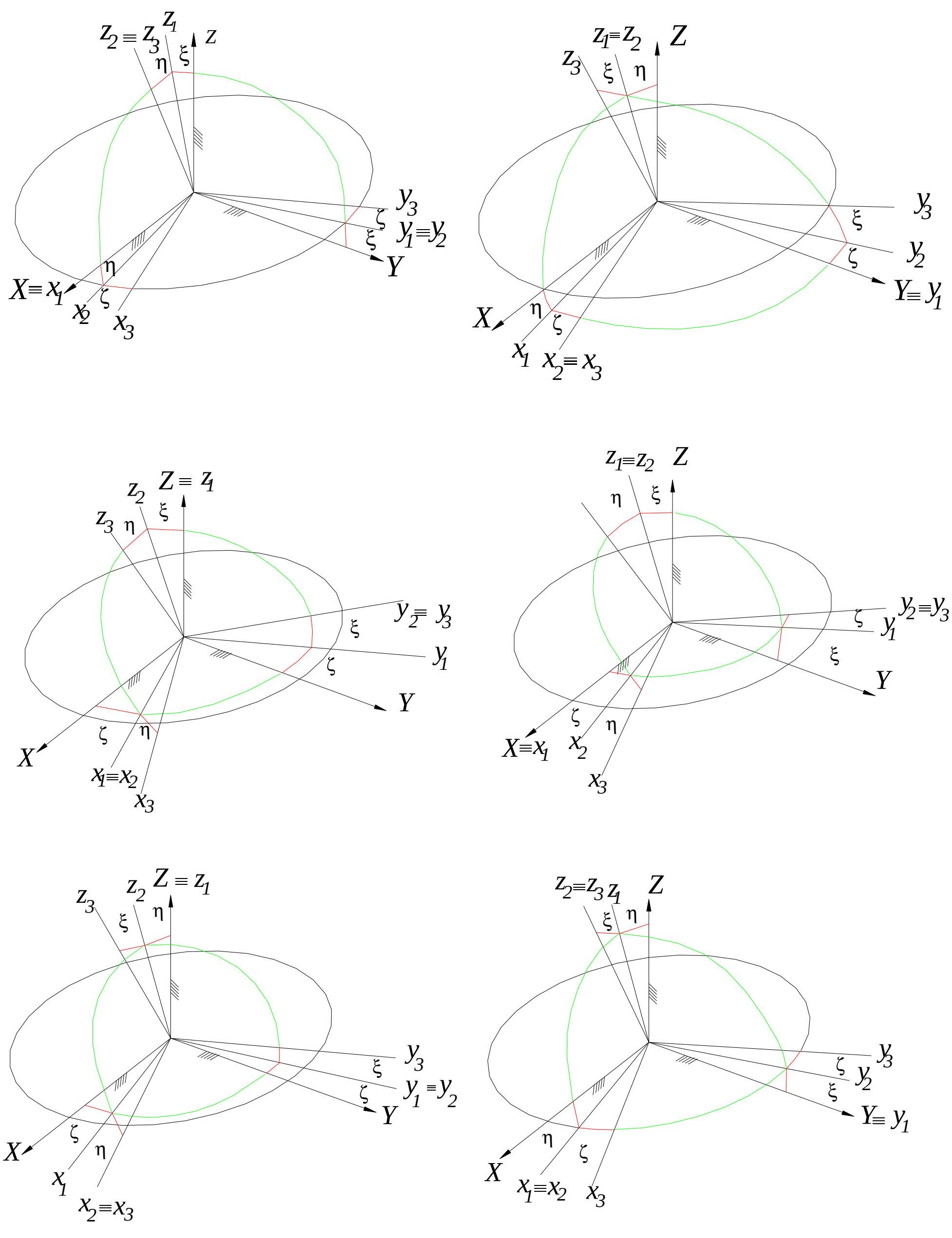

Figure 1. Nomencature of angles

3.1 Euler angles

Likely this is one of the oldest approach for the description of the body attitude. The angles are defined through an ordered sequence of rotations. Euler angles involve one body coordinate axis twice. Six sets of Euler angles can be distinguished. However, we will focus only on the sequences involving two rotations about the z axis (see Table 1)

Table 1: Definition of Euler angles

Rotation	First kind	Second kind
1	$y - z_0$	$y - z_0$
2	$q - x_1$	$v - y_1$
3	$f - z_2$	$f - z_2$

It is not difficult to show that, respectively, the transition matrices are for the first kind of Euler angles:

$$[T]_0^3 = \begin{bmatrix} c_\Psi c_f - s_\Psi c_q s_f & s_\Psi c_f - c_\Psi c_q s_f & s_q s_f \\ -c_\Psi c_f - s_\Psi c_q c_f & -s_\Psi s_f - c_\Psi c_q s_f & s_q c_f \\ s_\Psi s_q & -c_\Psi s_q & c_q \end{bmatrix}$$

and for the second kind of angles:

$$[T]_0^3 = \begin{bmatrix} c_\Psi c_v c_f - s_\Psi s_f & s_\Psi c_v c_f + c_\Psi s_f & -s_v c_f \\ -c_\Psi c_v s_f - s_\Psi c_f & -s_\Psi c_v s_f + c_\Psi c_f & s_v s_f \\ c_\Psi s_v & s_\Psi s_v & c_v \end{bmatrix}$$

3.2 Cardan angles

The three rotations involved in the definition of the Cardan angles are those about each axis of the body fixed cartesian reference system.

Table 2. Definition of Cardan angles

Rotation	1st kind	2nd kind	3rd kind	4th kind	5th kind	6th kind
1	$x - x_0$	$h - y_0$	$z - z_0$	$x - x_0$	$z - z_0$	$h - y_0$
2	$h - y_1$	$z - z_1$	$x - x_1$	$z - z_1$	$h - y_1$	$x - x_1$
3	$z - z_2$	$x - x_2$	$h - y_2$	$h - y_2$	$x - x_2$	$z - z_2$

As shown in Table 2, six types of Cardan angles can be enumerated. Thus, the transition matrix, for each enumerated set, is reported in the following:

- 1st kind of Cardan angles

$$[T]_0^3 = \begin{bmatrix} c_h c_z & c_x s_z + s_x s_h c_z & s_x s_z - c_x s_h c_z \\ -c_h s_z & c_x c_z - s_x s_h s_z & s_x c_z + c_x s_h s_z \\ s_h & -s_x c_h & c_x c_h \end{bmatrix}$$

- 2nd kind of Cardan angles

$$[T]_0^3 = \begin{bmatrix} c_z c_h & s_z & -s_h c_z \\ -c_x c_h s_z + s_x s_h & c_x c_z & c_x s_h s_z + s_x c_h \\ s_x c_h s_z + c_x s_h & -s_x c_z & -s_x s_h s_z + c_x c_h \end{bmatrix}$$

- 3rd kind of Cardan angles

$$[T]_0^3 = \begin{bmatrix} c_h c_z - s_x s_h s_z & c_h s_z + s_x s_h c_z & -c_x s_h \\ -c_x s_z & c_x c_z & s_x \\ s_h c_z + s_x c_h s_z & s_h s_z - s_x c_h c_z & c_x c_h \end{bmatrix}$$

- 4th kind of Cardan angles

$$[T]_0^3 = \begin{bmatrix} c_h c_z & c_x c_h s_z + s_x s_h & s_x c_h s_z - c_x s_h \\ -s_z & c_x c_z & s_x c_z \\ s_h c_z & c_x s_h s_z - s_x c_h & s_x s_h s_z + c_x c_h \end{bmatrix}$$

- 5th kind of Cardan angles

$$[T]_0^3 = \begin{bmatrix} c_h c_z & c_h s_z & -s_h \\ -c_x s_z + s_x s_h c_z & c_x c_z + s_x s_h s_z & s_x c_h \\ s_x s_z + c_x s_h c_z & -s_x c_z + c_x s_h s_z & c_x c_h \end{bmatrix}$$

- 6th kind of Cardan angles

$$[T]_0^3 = \begin{bmatrix} c_h c_z + s_x s_h s_z & c_x s_z & -s_h c_z + s_x c_h s_z \\ -c_h s_z + s_x s_h cz & c_x c_z & s_h s_z + s_x c_h c_z \\ c_x s_h & -s_x & c_x c_h \end{bmatrix}$$

In the technical literature, the angles x, h and z are usually referred as *YAW*, *PITCH* and *ROLL* angles. Often the system {*yaw*-*pitch*-*roll*} (5th kind of Cardan angles) is used in vehicle dynamics.

3.3 Euler parameters

Euler demonstrated that the spherical rigid displacement of a body could be obtained by rotating the body about a fixed axis. Let $\{u\} = \{u_x \quad u_y \quad u_z\}^T$ be the versor of the positive direction of such rotation axis, q the angle of rotation, then the Euler parameters[1] are defined as follows:

$e_0 = \cos\frac{q}{2}, e_1 = u_x \sin\frac{q}{2}, e_2 = u_y \sin\frac{q}{2}, e_3 = u_z \sin\frac{q}{2},$ with $e_0^2 + e_1^2 + e_2^2 + e_3^2 = 1$ normalizing condition.

The transform matrix has the following form:

$$[T] = 2\begin{bmatrix} e_0^2 + e_1^2 - \frac{1}{2} & e_1 e_2 - e_0 e_3 & e_1 e_3 + e_0 e_2 \\ e_1 e_2 + e_0 e_3 & e_0^2 + e_2^2 - \frac{1}{2} & e_2 e_3 - e_0 e_1 \\ e_1 e_3 - e_0 e_2 & e_2 e_3 + e_0 e_1 & e_0^2 + e_3^2 - \frac{1}{2} \end{bmatrix}$$

The matrix can be expressed as the product of two matrices $[T] = [E][G]^T$ where

$$[E] = \begin{bmatrix} -e_1 & e_0 & -e_3 & e_2 \\ -e_2 & e_3 & e_0 & -e_1 \\ -e_3 & -e_2 & e_1 & e_0 \end{bmatrix}, \text{ and } \quad [G] = \begin{bmatrix} -e_1 & e_0 & e_3 & -e_2 \\ -e_2 & -e_3 & e_0 & e_1 \\ -e_3 & e_2 & -e_1 & e_0 \end{bmatrix}$$

[1] In 1770 Euler decided to generate through algebra some *magic squares* whose properties are identical to those of matrix $[T]$. He demonstrated that he could parametrize the elements with angles today known as *Euler angles*. He was not very satisfied of his discovery because of the presence of trigonometric functions. Thus he looked for a parametrization involving four parameters only. He must have reached the result in an almost divinatory manner. In fact, he never disclosed the steps involved in his derivations.

3.4 Rotation axis and angle of rotation

Given the versor $\{u\} = \{u_X \quad u_Y \quad u_Z\}^T$ of the finite rotation axis and the rotation angle, the transition matrix takes the form

$$[T] = \begin{bmatrix} u_X^2 v_y + c_y & u_X u_Y v_y + u_Z s_y & u_X u_Z v_y - u_Y s_y \\ u_X u_Y v_y - u_Z s_y & u_Y^2 v_y + c_y & u_Y u_Z v_y + u_X s_y \\ u_X u_Z v_y + u_Y s_y & u_Y u_Z v_y - u_X s_y & u_Z^2 v_y + c_y \end{bmatrix}$$

where $v_\Psi = 1 - \cos y$, $c_y = \cos y$, $s_y = \sin y$.

3.5 Cayley-Klein parameters

The Cailey-Klein parameters, in terms of Euler angles of the first kind, are defined as follows

$$a = e^{i(y+f)/2} \cos\frac{q}{2};\ b = ie^{i(f-y)/2} \sin\frac{q}{2}; g = ie^{-i(f-y)/2} \sin\frac{q}{2}; d = e^{-i(y+f)/2} \cos\frac{q}{2}$$

where $ad - bg = 1$ Thus, the transition matrix takes the form

$$[T] = \begin{bmatrix} \frac{1}{2}\left(a^2 - b^2 - g^2 - d^2\right) & \frac{1}{2}\left(g^2 - a^2 + d^2 - b^2\right) & gd - ad \\ \frac{1}{2}\left(a^2 + g^2 - b^2 - d^2\right) & \frac{1}{2}\left(a^2 + g^2 + b^2 + d^2\right) & -i(ab + gd) \\ bd - ag & i(ag + bd) & ad + bg \end{bmatrix}$$

4. Transform formulas

This section summarizes useful relationships to transform one set of body parameters into another set.

4.1 From Euler angles to Euler parameters

$$e_0 = \cos\frac{q}{2}\cos\frac{f+y}{2}\ ;\quad e_1 = sin\frac{q}{2}\cos\frac{y-f}{2}\ ;\quad e_2 = sin\frac{q}{2}\,sin\frac{y-f}{2}\ ;$$

$$e_3 = \cos\frac{q}{2}\,sin\frac{f+y}{2}$$

4.2 From Euler parameters to Euler angles

$$\tan y = \frac{e_1 e_3 + e_0 e_2}{e_0 e_1 - e_2 e_3},\ 0 \le y \le 2p\ ;\ \tan q = \frac{2\sqrt{e_1^2 + e_2^2}\sqrt{e_0^2 + e_3^2}}{e_0^2 - e_1^2 + e_2^2 + e_3^2},\ 0 \le q \le 2p\ ;\ \tan f = \frac{e_1 e_3 - e_0 e_2}{e_0 e_1 - e_2 e_3},\ 0 \le f \le 2p$$

4.3 From Euler parameters to Cayley-Klein parameters

$$a = e_0 + ie_1,\ b = e_2 + ie_1, g = -e_2 + ie_1, d = e_0 + ie_3$$

5. Inversion formulas

These formulas allow to obtain the parameters describing the body attitude from the knowledge of the rotation or transition matrix $[T]$. Regarding this operation, the following theorem appears useful \cite{beggs}: the orientation of body 1 relative to body 2 is uniquely specified by stating the values of three elements of $[T]$ which lie in any two rows, and one of four possible values of a fourth element, chosen so that the four elements do not lie in the same minor, and less than three elements in a row. The word “row” may be replaced by the word “column” throught the theorem.

5.1 Euler angles

Euler angles of the first kind, can be deduced from the elements of $[T]$ by means of the following expressions:

$\cos J = t_{33}, sin J = \pm\sqrt{1-\cos^2 J}$; $\cos y = \frac{t_{32}}{sin J}, sin y = -\frac{t_{31}}{sin J}$; $\cos f = \frac{t_{23}}{sin J}, sin f = \frac{t_{23}}{sin J}$,

There is an indeterminacy for $J = 0$.

5.2 Cardan angles

Cardan angles of the first kind, can be deduced from the elements of $[T]$ by means of the following expressions:

$$sin h = t_{31}; \cos h = \pm\sqrt{1-t_{31}^2}; sin z = -\frac{t_{21}}{\cos h}; \cos z = \frac{t_{11}}{\cos h}$$

$$\cos x = \frac{t_{33}}{\cos h}; sin x = -\frac{t_{32}}{\cos h}$$

there is an indetermacy for $h = \frac{p}{2}$

5.3 Euler parameters

Adopting the Euler parameters, indeterminacy can be avoided by choosing, among those reported in Table 3, the appropriate set of equations.

Table 3. Inversion Formulas for Eulers parameters

	e_0 ? 0	e_1 ? 0	e_2 ? 0	e_3 ? 0
e_0	$\pm\frac{\sqrt{1+t_{11}-t_{22}-t_{33}}}{2}$	$\frac{t_{12}+t_{21}}{4e_1}$	$\frac{t_{13}+t_{31}}{4e_2}$	$\frac{t_{23}-t_{32}}{4e_3}$
e_1	$\frac{t_{12}+t_{21}}{4e_0}$	$\pm\frac{\sqrt{1-t_{11}+t_{22}-t_{33}}}{2}$	$\frac{t_{23}+t_{32}}{4e_2}$	$\frac{t_{31}-t_{13}}{4e_3}$
e_2	$\frac{t_{13}+t_{31}}{4e_0}$	$\frac{t_{23}+t_{32}}{4e_1}$	$\pm\frac{\sqrt{1-t_{11}-t_{22}+t_{33}}}{2}$	$\frac{t_{12}-t_{21}}{4e_3}$
e_3	$\frac{t_{23}+t_{32}}{4e_0}$	$\frac{t_{31}-t_{13}}{4e_1}$	$\frac{t_{12}-t_{21}}{4e_2}$	$\pm\frac{\sqrt{1+t_{11}+t_{22}+t_{33}}}{2}$

5.4 Rotation axis and angle of rotation

The angle f of rotation is computed by means of the formula $f = \cos^{-1}\left(\frac{t_{11}+t_{22}+t_{33}-1}{2}\right)$, whereas, for $f \neq p$, the cartesian components of the rotation axis follow from

$$u_X = \frac{t_{32}-t_{23}}{2 sin f}, u_Y = \frac{t_{13}-t_{31}}{2 sin f}, u_Z = \frac{t_{21}-t_{12}}{2 sin f}$$

6. Transition matrices for velocity components

Newton-Euler equations are expressed in terms of the angular velocity components. Thus, these should be expressed as a function of the derivatives of the body attitude parameters. The angular velocity components (w_x, w_y, w_z) are those in body fixed coordinates, whereas (w_X, w_Y, w_Z) are those in the fixed reference system. The components are related by the following transform

$$\begin{Bmatrix} w_X \\ w_Y \\ w_Z \end{Bmatrix} = [\mathrm{T}] \begin{Bmatrix} w_x \\ w_y \\ w_z \end{Bmatrix}$$

1. Euler angles
   - 1st kind

$$\begin{Bmatrix} w_x \\ w_y \\ w_z \end{Bmatrix} = \begin{bmatrix} \sin q \sin f & \cos f & 0 \\ \sin m \cos f & -\sin f & 0 \\ \cos q & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot{y} \\ \dot{q} \\ \dot{f} \end{Bmatrix}$$

   - 2nd kind

$$\begin{Bmatrix} w_x \\ w_y \\ w_z \end{Bmatrix} = \begin{bmatrix} -\sin n \cos f & \cos f & 0 \\ \sin m \cos f & -\sin f & 0 \\ \cos n & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot{y} \\ \dot{q} \\ \dot{f} \end{Bmatrix}$$

2. Cardan angles

   - 1st kind

$$\begin{Bmatrix} w_x \\ w_y \\ w_z \end{Bmatrix} = \begin{bmatrix} \cos z & 0 & 0 \\ -\cos x sinz & 1 & 0 \\ sinz sinx & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{h} \\ \dot{z} \end{Bmatrix}$$

   - 2nd kind

$$\begin{Bmatrix} w_x \\ w_y \\ w_z \end{Bmatrix} = \begin{bmatrix} 1 & \sin z & 0 \\ 0 & \cos x \cos z & \sin x \\ 0 & -\sin x \cos z & \cos x \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{h} \\ \dot{z} \end{Bmatrix}$$

   - 3rd kind

$$\begin{Bmatrix} w_x \\ w_y \\ w_z \end{Bmatrix} = \begin{bmatrix} \cos h & 0 & -\cos x \sin h \\ 0 & 1 & \sin x \\ \sin h & 0 & \cos x \cos h \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{h} \\ \dot{z} \end{Bmatrix}$$

   - 4th kind

$$\begin{Bmatrix} w_x \\ w_y \\ w_z \end{Bmatrix} = \begin{bmatrix} \cos h \cos z & 0 & -\sin h \\ -\sin z & 1 & 0 \\ \sin h \cos z & 0 & \cos h \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{h} \\ \dot{z} \end{Bmatrix}$$

   - 5th kind

$$\begin{Bmatrix} w_x \\ w_y \\ w_z \end{Bmatrix} = \begin{bmatrix} 1 & 0 & -sinh \\ 0 & \cos x & sinx \cos h \\ 0 & -sinx & \cos x \cos h \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{h} \\ \dot{z} \end{Bmatrix}$$

- 6^{th} kind

$$\begin{Bmatrix} w_x \\ w_y \\ w_z \end{Bmatrix} = \begin{bmatrix} \cos z & \cos x \sin z & 0 \\ -\sin z & \cos x \cos z & 0 \\ 0 & -\sin x & 1 \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{h} \\ \dot{z} \end{Bmatrix}$$

3. Euler parameters

$$\{w\}^{(0)} = 2[E]_i^0 \{r\}, \{r\} = \tfrac{1}{2}\left[E^T\right]_i^0 \{w\}^{(0)}; \{w\}^{(0)} = 2[G]_i^0 \{r\}, \{r\} = \tfrac{1}{2}\left[G^T\right]_i^0 \{w\}^{(0)}$$

$$\{w\}^{(0)} = \{0 \quad w_x \quad w_y \quad w_z\}^T, \{w\}^{(i)} = \{0 \quad w_x \quad w_y \quad w_z\}^T, \{p\} = \{e_0 \quad e_1 \quad e_2 \quad e_3\}^T$$

7. Conclusions

In this paper have been summarized the classical methods to specify the attitude of a rigid body, including some ad hoc derived analytical expression. These methods can be used in many fields of mechanics as well as in computer graphics applications, where there is the need of simulating the movements of objects.

In particular the proposed formulas can be used to link equations from different models or compare results from different approaches. They can work as useful recipes for every designer or engineer involved in kinematic or dynamic analysis.

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