INTERNATIONAL DESIGN CONFERENCE - DESIGN 2004
Dubrovnik, May 18 - 21, 2004.

DESIGN 2004

MATHEMATICAL ANALYSIS OF INFORMATION FLOWS IN DESIGN AND EDUCATION

H. Miki, Y. Kakuda and M. Kikuchi

Keywords: design theory, education, information flow, normativeness

1. Introduction

Education seems to be closely parallel to design. In design activities, we try to seek a solution of design by which we can produce entities according to specifications. Similarly in educational activities, we try to plan an educational practice which enables students to behave according to educational objectives. Actually T. Monnai tried to introduce various design methods into computer aided instructions (Monnai, 1992).
In this paper, we mathematically formulate education within a framework of mathematical theory of design; and we show that design and education have the structure in common in our formulation.
We use the framework of Abstract Design Theory (ADT) for formulating education. ADT is introduced and developed by Y. Kakuda as a mathematical theory of design based on General Design Theory pioneered by H. Yoshikawa (Yoshikawa, 1981) and Channel Theory, a theory of information flows (Barwise and Seligman, 1997). In ADT an information flow is represented by using and developing concepts of classifications, infomorphisms, channels which are defined in Channel Theory. We regard design as an information flow from our desires or needs to the physical world, and argue in terms of two classifications: one represents our desires or needs and another represents the physical world as its entities and their behaviours.
We give mathematical formulation of education within the framework of ADT and we show that we can formally argue education from the viewpoint of an information flow. By focusing on normative methods of education in our formulation, we try to clarify the role of normative aspects of education, which is concerned to the effectiveness of education. And then, we show that an information flow in education must be regarded as a normative part of it.
Usually designing is getting a solution to fulfil specifications. By comparing design with education in our formulation, we can presume that we suppose a sort of a normative relation between solutions and specifications, as in education. As a result, because of the resemblance of design and education in structure, we could bring normative aspects to design activities as a key to analyze design methods, and conversely we could bring various design methods or viewpoints of design theory to educational practices.

2. Representation of an information flow

We show representation of information flows with a concept of an information link which is newly defined in ADT with concepts of a classification, an infomorphism and a channel which are defined in Channel Theory. In ADT, information is represented by a binary relation at a classification, and an information flow is represented by a set of information links between classifications. Furthermore a set of information links is represented by a channel called its cover. We define these concepts in this section.

Information has a form that "a medium a carries an information content α". Thus, we consider a set tok(A) of media, a set typ(A) of information contents, and a binary relation $a \models_A \alpha$ between tok(A) and typ(A), meaning that a medium a carries an information content α. Mathematically we define a classification as follows. A *classification* $A = <$ tok(A), typ(A), $\models_A >$ consists of (1) a set tok(A) of objects to be classified, called the tokens of A, (2) a set typ(A) of objects used to classify the tokens, called the types of A, and (3) a binary relation $\models_A$ between tok(A) and typ(A).

$$\begin{tikzcd} \mathrm{typ}(A) \arrow[r, "f^\wedge"] \arrow[d, dash, "\models_A"] & \mathrm{typ}(B) \arrow[d, dash, "\models_B"] \\ \mathrm{tok}(A) & \mathrm{tok}(B) \arrow[l, "f^\vee"] \end{tikzcd} \qquad \begin{tikzcd} & & C & \\ A_1 \arrow[urr, "f_1"] & A_2 \arrow[ur, "f_2"] & \cdots & A_n \arrow[ul, "f_n"'] \end{tikzcd}$$

Figure 1. An infomorphism and a channel

An infomorphism between two classifications is a contra-variant pair of maps which satisfy the bi-conditional property of classification relations.

Let A and B be classifications and $f^\wedge$: typ(A) → typ(B) and $f^\vee$: tok(A) → tok(B) be maps. $f = < f^\wedge, f^\vee >$ is said to be an *infomorphism* and write $f : A \to B$ if the following condition is satisfied: $f^\vee(b) \models_A \alpha$ if and only if $b \models_B f^\wedge(\alpha)$ for any $\alpha \in$ typ(A) and for any $b \in$ tok(B), which is called *fundamental property*. We will omit superscripts $^\wedge$ and $^\vee$ if no confusion is likely.

Given a family $L = \{ A_\lambda \}_{\lambda \in \Lambda}$ of classifications, a *channel* $\mathbf{C} = \{ f_\lambda : A_\lambda \to C \}_{\lambda \in \Lambda}$ for L is defined as a family of infomorphisms which has a common co-domain classification C, called the *core* of a channel, and all domains of infomorphisms in $\mathbf{C}$ are in L. A channel shown in Figure 1 is $\{ f_\lambda : A_\lambda \to C \}_{\lambda \in \Lambda}$ where $\Lambda = \{1, 2, \cdots, n\}$.

Let $\mathbf{C} = \{ f_\lambda : A_\lambda \to C \}_{\lambda \in \Lambda}$ and $\mathbf{D} = \{ f'_\lambda : A_\lambda \to D \}_{\lambda \in \Lambda}$ be channels with the same component classifications $\lambda \in \Lambda$. A *refinement infomorphism* r from $\mathbf{C}$ to $\mathbf{D}$ is an infomorphism $r : C \to D$ such that $r^\wedge(f_\lambda^\wedge(\alpha)) = f'^\wedge_\lambda(\alpha)$ for all $\alpha \in$ typ(A_λ) and $f_\lambda^\vee(r^\vee(d)) = f'^\vee_\lambda(d)$ for all $d \in$ tok(D).

The channel $\mathbf{C}$ is a refinement of the channel $\mathbf{D}$ if there is a refinement infomorphism r from $\mathbf{C}$ to $\mathbf{D}$.

Now we define an information link in order to express an information flow by a set of them. Let A and B be classifications. Whenever an information flow from A to B undergoes a metamorphosis from an information $a \models_A \alpha$ to an information $b \models_B \beta$, it should satisfies that $a \models_A \alpha$ implies $b \models_B \beta$. Here an *information link* $< R, S >$ from A to B is defined as follows:

Let A and B be classifications. A pair of correspondences R : typ(A)→ typ(B) and S : tok(B)→ tok(A) is said to be a contra-variant pair $< R, S >$ from A to B, written $< R, S >: A \to B$. A contra-variant pair $< R, S >$ is said to be an *information link* if the following condition is satisfied.

$$\forall \alpha \forall \beta \forall a \forall b \, (\alpha R \beta \wedge b S a \Rightarrow (a \models_A \alpha \Rightarrow b \models_B \beta)).$$

$$\begin{tikzcd} \mathrm{typ}(A) \arrow[r, bend left, "R"'] \arrow[d, dash, "\models_A"] & \mathrm{typ}(B) \arrow[d, dash, "\models_B"] \\ \mathrm{tok}(A) & \mathrm{tok}(B) \arrow[l, bend left, "S"'] \end{tikzcd}$$

Figure 2. An information link

Let $L = \{ A_\lambda \}_{\lambda \in \Lambda}$ be a family of classifications. An information flow is represented by a set F of information links in L, and can be represented by a cover of F in L. In order to define a cover of

F in L, we must define a type-ordered classification and an ordered channel as follows: A type-ordered classification A is a classification whose type set is a partially ordered set with an order $\leq_A$ such that $\alpha \leq_A \beta \Rightarrow \forall\ a \in$ tok(A)($a \models_A \alpha \Rightarrow a \models_A \beta$). A family **C** of infomorphisms is said to be an ordered channel for L if the core of **C** is a type-ordered classification.

Let **C** $= \{f_\lambda : A_\lambda \to C\}_{\lambda\in\Lambda}$ be an ordered channel for $L = \{ A_\lambda \}_{\lambda\in\Lambda}$. We define **C** *covers* $<R,S>: A_\lambda \to A_\mu$ if $\alpha R \beta$ implies $f_\lambda(\alpha) \leq_C f_\mu(\beta)$ for all $\alpha \in$ typ(A_λ) and $\beta \in$ typ(A_μ), and $f_\mu(c)\, S\, f_\lambda(c)$ for all $c \in$ tok(C). We say that **C** *covers* a set F of information links, if **C** *covers* $<R,S>$ for all $<R,S> \in F$. **C** is called a *cover* of F if it covers a set F . **C** is a *minimal cover* of $<R,S>$ if it covers $<R,S>$ and for every other ordered channel **D**= $\{g_\lambda : A_\lambda \to D\}_{\lambda\in\Lambda}$ covering $<R,S>$ there exists a unique order- preserving infomorphism from C to D. The core of a minimal cover of F represents regulation of information flows expressed in F as a whole.

In order to see the core of a minimal cover of a set F of information links in a family L of classifications, we must define a path, a propagation path, and a dual semi-invariant $<R_F, S_F>$.

A *path* for L is a map $s : \Lambda \to \bigcup_{\lambda\in\Lambda}$ tok(A_λ) such that $s(\lambda) \in$ tok(A_λ) for each $\lambda \in \Lambda$. Let F be a set of information links in L. A *propagation path* for F is a path s such that $s(\mu)\, S\, s(\lambda)$ for every $<R,S>: A_\lambda \to A_\mu$ in F . We define the sum $\sum_{\lambda\in\Lambda} A_\lambda$ of $L = \{A_\lambda\}_{\lambda\in\Lambda}$ as follows: (1) tok($\sum_{\lambda\in\Lambda} A_\lambda$) $= \prod_{\lambda\in\Lambda}$ tok(A_λ) = $\{ s \mid s$ is a path for $F \}$, (2) typ($\sum_{\lambda\in\Lambda} A_\lambda$) = $\sum_{\lambda\in\Lambda}$ typ(A_λ) = $\{<\alpha,\lambda> \mid \lambda \in \Lambda,\ \alpha \in$ typ(A_λ)$\}$, and (3) $s \models_{\sum_{\lambda\in\Lambda} A_\lambda} <\alpha,\lambda>$ if and only if $s_\lambda \models_{A_\lambda} \alpha$.

Now we define a *dual semi-invariant* $<R_F, S_F>$ for F in L on $\sum_{\lambda\in\Lambda} A_\lambda$ as follows: (1) $<\alpha,\lambda> R_F <\beta,\mu>$ if and only if there exist a link $<R,S>: A_\lambda \to A_\mu$ in F such that $\alpha R \beta$, and (2) $S_F = \{ s \in$ tok($\sum_{\lambda\in\Lambda} A_\lambda$) $\mid s$ is a propagation path for $F \}$.

We can prove that there exists a pair of a type-ordered classification C_M and an infomorphism f from $\sum_{\lambda\in\Lambda} A_\lambda$ to C_M for $<R_F, S_F>$ with the property $\alpha R \beta$ implies $f(\alpha) \leq_{C_M} f(\beta)$ and $f(c) \in S_F$ for $c \in$ tok(C_M). A type-ordered classification C_M can be defined as follows: (1) tok(C_M)= S_F , (2) typ(C_M)=typ($\sum_{\lambda\in\Lambda} A_\lambda$) / $\sim_{R_F}$, where let $\leq_{R_F}$ be the least preordering on typ($\sum_{\lambda\in\Lambda} A_\lambda$), and let $\sim_{R_F}$ be a equivalence relation defined by $\alpha \sim_{R_F} \alpha'$ if and only if $\alpha \leq_{R_F} \alpha'$ and $\alpha' \leq_{R_F} \alpha$, (3) $s \models_{C_M} [<\alpha,\lambda>]_{\sim_{R_F}}$ if $s \models_{\sum_{\lambda\in\Lambda} A_\lambda} <\alpha,\lambda>$, and (4) $[<\alpha,\lambda>]_{\sim_{R_F}} \leq_{C_M} [<\beta,\mu>]_{\sim_{R_F}}$ if $<\alpha,\lambda> \leq_{R_F} <\beta,\mu>$. By using this pair of C_M and f , we can prove the following lemma. This type-ordered classification C_M is the core of a minimal cover **C** of F in L .

Minimal Covering Lemma. Let F be a set of information links in $L = \{ A_\lambda \}_{\lambda\in\Lambda}$. There exists an ordered channel **C**= $\{ f_\lambda : A_\lambda \to C \}_{\lambda\in\Lambda}$ with the following properties: **C** covers F and If **D**= $\{ f'_\lambda : A_\lambda \to D \}_{\lambda\in\Lambda}$ is another ordered channel such that **D** covers F , then there exists a unique order-preserving refinement infomorphism $r : C \to D$.

3. A design system

We show a design system by using the notion of an information flow. Design can be regarded as an information flow from our desires or needs to the physical world. In ADT, design is argued in terms of two classifications: one represents our desires or needs and another represents the physical world as its entities with their behaviours.

In order to represent our desire or needs we use the classification generated by a *regular theory* defined in Channel Theory (Barwise and Seligman, 1997). A regular theory is a generalization of the classical theory of propositional logic. Let Σ be a set. A sequent of Σ is an ordered pair $<\Gamma, \Delta>$ of subsets Γ, Δ of Σ. A binary relation Θ between subsets of Σ is called a Gentzen consequence relation on Σ. A theory is a pair $T = <\Sigma, \Theta>$, where Σ is called a set of types of T. A partition of a set Σ is a pair $<\Gamma, \Delta>$ of subsets of Σ such that $\Gamma \cup \Delta = \Sigma$ and $\Gamma \cap \Delta = \phi$.

Let $T = <\Sigma, \Theta>$ be a regular theory. We say a sequent $<\Gamma, \Delta>$ of Σ is T-consistent if $<\Gamma, \Delta> \notin \Theta$. The classification Cla(T) generated by T is the classification whose tokens are the T-consistent partitions $<\Gamma, \Delta>$ of Σ, and whose types are the types of T, and the classification relation is defined by $<\Gamma, \Delta> \models_{Cla(T)} \alpha$ if and only if $\alpha \in \Gamma$. A T-consistent partitions $<\Gamma, \Delta>$ of Σ can be regarded as an ideal token of the classification generated by T.

If we consider a set Σ of our desire or needs, then the regular theory $T = <\Sigma, \Theta>$ could arise. We can regard a T-consistent sequent $<\Gamma, \Delta>$ as a specification, in which $\alpha \in \Gamma$ is positive and $\beta \in \Delta$ is negative, and then an ideal or complete specification can be expressed by a token of Cla(T). And we consider the classification B, whose tokens are entities, whose types are their behaviours, and whose binary relation is defined by $b \models_{\boldsymbol{B}} \beta$ if and only if b behaves in β.

We try to seek a solution of design by which we can produce entities according to our desire or needs. On one hand, even as we have desires or needs, we may positively suppose that there exists a set I = $\{<R_I, S_I>, <R_I^{-1}, S_I^{-1}>\}$ of information links between these two classifications which represents an idealistic correspondence between them although we do not clearly know what it is. We call the set I of information links the *ideal production*. On the other hand, our design activity is represented by information links between classifications including these two classifications in production. By Minimal Covering Lemma, we can mathematically show that there is an order-preserving refinement infomorphism from the cover of the ideal production to the cover of the set of these information links representing our design activities when we appropriately design products to fulfil our desires or needs.

4. Mathematical formulation of education

Now we show mathematical formulation of education within the framework of ADT. By means of our educational practices, we expect students behave according to fulfilling our educational objectives. In the first place, we have to fix two classifications. One is Cla(T_{Obj}) generating from a regular theory $T_{Obj} = <\Sigma_{Obj}, \Theta_{Obj}>$ in which we express our educational objectives as a regular theory $T = <\Sigma, \Theta>$ for desire or needs in the design system. Another one is a classification B_{St} expressing the students with their behaviours as an object of education: tok(B_{St}) is a set of instances of students at various times, typ(B_{St}) is a set of their behaviours, and $b \models_{\boldsymbol{B}} \beta$ means that b behave in β.

As in the ideal production, we may positively suppose that there exists a set I_{Edu} of information links between these two classifications, called the *ideal education*, which represents an idealistic correspondence between them although we do not clearly know what it is. By Minimal Covering Lemma, we have the minimal cover $\mathbf{C}_I = \{ f_I : \text{Cla}(T_{Obj}) \to C_I, g_I : B_{St} \to C_I \}$ for I_{Edu}. Then a type of the core C_I of the cover $\mathbf{C}_I$ can be regarded as a *function* of education induced by an educational objective α and a behaviour β because $f_I(\alpha) = g_I(\beta)$ for $\alpha \in \Sigma_{Obj}$ and $\beta \in$ typ(B_{St}), a token c of the core C_I can be regarded as an *image* which realizes an instance $b \in$ tok(B_{St}) of a student and forms a concept $<\Gamma, \Delta> \in$ tok(Cla(T_{Obj})) of a student concerning educational objectives because $<\Gamma, \Delta> = f_I(c)$ and $b = g_I(c)$. In the next place, in order to fulfil educational objectives, we define some classifications with Cla(T_{Obj}) and B_{St}, and construct a set F_{Edu} of information links between these classifications, and then planning educational practices is getting the minimal cover $\mathbf{C}_{Edu}$ of F_{Edu}, which also covers I_{Edu} and includes f_{Edu} :Cla(T_{Obj}) $\rightarrow C_{Edu}$, $g_{Edu} : B_{St} \rightarrow C_{Edu}$ A token of the core C_{Edu} can be regarded as an *educational plan* and a type of the core C_{Edu} can be regarded as an *attribute* of elements in those educational practices. By Minimal Covering Lemma, there exists the refinement infomorphism r from $\mathbf{C}_I$ to $\mathbf{C}_{Edu}$ because $\mathbf{C}_I$ is the minimal cover of I_{Edu}, if $\mathbf{C}_{Edu}$ covers I_{Edu}, that is, planned educational practices are effective in fulfilling educational objectives. For an element of educational objective $\alpha \in \Sigma_{Obj}$, the attribute $f_{Edu}(\alpha)$ obtained by interpreting α through the educational practice C_{Edu} is equivalent to the attribute $r(f_I(\alpha))$ required by the function $f_I(\alpha)$ which is induced by α, and similarly for a behaviour $\beta \in$ typ(B_{St}), the attribute $g_{Edu}(\beta)$ obtained by interpreting β through the educational practice C_{Edu} is equivalent to the attribute $r(g_I(\beta))$ required by the function $g_I(\beta)$ which is induced by β. For an educational plan $c_{Edu} \in$ tok(C_{Edu}), the concept $f_{Edu}(c_{Edu})$ is equivalent to the concept $f_I(r(c_{Edu}))$ formed by the image $r(c_{Edu})$ which is imaged by c_{Edu}, and the instance of the student $g_{Edu}(c_{Edu})$ which is formed through the educational practice based on c_{Edu} is equivalent to the instance $g_I(r(c_{Edu}))$ realized by the image $r(c_{Edu})$ which is imaged by c_{Edu}.

$$\begin{tikzcd}
 & \mathrm{typ}(C_{Edu}) & \\
 & \mathrm{typ}(C_I) \arrow[u, "r"] & \\
\mathrm{typ}(\mathrm{Cla}(T_{Obj})) \arrow[uur, "f_{Edu}"] \arrow[ur, "f_I"'] & & \mathrm{typ}(B_{St}) \arrow[uul, "g_{Edu}"'] \arrow[ul, "g_I"]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathrm{tok}(C_{Edu}) \arrow[ddl, "f_{Edu}"'] \arrow[d, "r"] \arrow[ddr, "g_{Edu}"] & \\
 & \mathrm{tok}(C_I) \arrow[dl, "f_I"] \arrow[dr, "g_I"'] & \\
\mathrm{tok}(\mathrm{Cla}(T_{Obj})) & & \mathrm{tok}(B_{St})
\end{tikzcd}$$

Figure 3. Mathematical formulation of education

5. Normative methods in education and information flows

We focus on two types of the educational practices, one is normative approach and another is constructive or interpretive one, in order to discuss about normativeness in education.

On one hand, we show that we can express the educational method using the revised version of Bloom's Taxonomy (Anderson, 2001), which has normative approach, in an information flow in our formulation of education.

The Taxonomy Table is the two-dimensional table: the columns of the table consist of categories for cognitive process, the rows of the table consist of categories for knowledge, and the cells of the table are where the knowledge and cognitive process dimensions intersect. A description of plans of educational practices is called a curriculum unit. A curriculum unit divided into three components: objectives, instructional activities, and assessments. The way to analyze and plan a curriculum unit by using the Taxonomy Table is described as follows. The statements of each component of a curriculum unit are classified in the Taxonomy Table after re-reading and re-examination for a placement of better guesses, and then we get three separate Taxonomy Tables: one for the statement of objectives, one for the instructional activities and one for the assessments. Finally, the consistency across the three tables is examined, comparing the classifications of three components. In our formulation of education, classifying descriptions of a curriculum unit in the cells of the Taxonomy Table can be represented by making classifications defined in this paper, and the method for analyzing a curriculum unit can be represented by information links between these classifications (Miki, et al., 2002). By the minimal covering lemma in this paper, we can get the minimal cover of a set of these information links and the refinement infomorphism from the cover $\mathbf{C}_I$ of the ideal education $\mathbf{I}_{Edu}$ in this educational practice to the minimal cover of a set of these information links, which represents the method for analyzing this curriculum unit. On the other hand, educational practices using the notion of *sociomathematical norms* is said to be effective in constructive approaches to education of mathematics. It is the normative aspects of mathematical discussions in a classroom that are specific to students' mathematical activity, for example, normative understandings of what counts as mathematically different, mathematically sophisticated, mathematically efficient , and mathematically elegant in a classroom (Yackel and Cobb, 1996). In our formulation of education, each sociomathematical norm can be represented by an information link between the classification of educational activities in the classroom and the classification of students with their behaviours. By the minimal covering lemma in this paper, we can also get the minimal cover of a set of these information links and the refinement infomorphism from the cover $\mathbf{C}_I$ of the ideal education in this educational practice to the minimal cover of a set of these information links, which represents sociomathematical norms.

In our educational framework we can analyze these two methods as follows: Bloom's method put emphasis on the classification relation on the classification of curriculum units, and Cobb's method put emphasis on the information links; Bloom's method put emphasis on the infomorphism from the classification of educational objectives to the core of its minimal cover, Cobb's method put emphasis on one from the classification of students to the core of its minimal cover.

6. Conclusions

We gave mathematical formulation of education within the framework of ADT and we found that we can formally argue education from the viewpoint of an information flow. Consequently we found that design and education have the structure in common in our formulation.

By focusing on normative methods of educational practices, we showed that we can represent both two examples, Bloom's method and Cobb's method, as an information flow in our formulation of education. We found that an information flow in education must be regarded as the normative part of it, which is concerned to the effectiveness of education. Moreover we found that the difference between normative and constructive types of approaches to educational practices depends on emphasis on normative part of them, and that these two types do not conflict. Usually designing is getting a solution to fulfil specifications. By comparing design with education in our formulation, we can presume that we suppose a sort of a normative relation between solutions and specifications. As a result, because of the resemblance of design and education in structure, we could bring normative aspects to design activities as a key to analyze design methods, and conversely we could bring various design methods or viewpoints of design theory to educational practices.

References

Anderson, L. W.,. "A Taxonomy for Learning, Teaching, and Assessing", Addison Wesley Longman, 2001.

Barwise, J., and Seligman, J., "Information Flow", Cambridge Tracts in Theoretical Computer Science 44, Cambridge University Press, 1997.

Miki, H., Kikuchi, M., Kakuda, Y. , "An abstract design model of educational activities using Bloom's taxonomy", Proceedings of IWES'02, 2002, pp 89-94.

Monnnai, T., "The Educational Use of a Computer and its Environmental Implications", Waseda review of education Japan, 1992, pp.1-15.

Yackel, E., Cobb, P., "Sociomathematical norms, argumentation, and autonomy in mathematics", Journal for Research in Mathematics Education Vol.27 No.4, 1996.

Yoshikawa, H., "General Design Theory and a CAD system", Man-Machine Communication in CAD/CAM North-Holland, 1981, pp 35–57.

Hirofumi Miki, Dr.
Kenmei Women's Junior College
Honmachi 68, Himeji, 6700012, Japan
Telephone: +81-792-23-8459, Telefax: +81-792-23-8478
E-mail: miki@kenmei.ac.jp